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Topology shapes dynamics of higher-order networks

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Higher-order networks capture the many-body interactions present in complex systems, shedding light on the interplay between topology and dynamics. The theory of higher-order topological dynamics, which combines higher-order interactions with discrete topology and nonlinear dynamics, has the potential to enhance our understanding of complex systems, such as the brain and the climate, and to advance the development of next-generation AI algorithms. This theoretical framework, which goes beyond traditional node-centric descriptions, encodes the dynamics of a network through topological signals—variables assigned not only to nodes but also to edges, triangles and other higher-order cells. Recent findings show that topological signals lead to the emergence of distinct types of dynamical state and collective phenomena, including topological and Dirac synchronization, pattern formation and triadic percolation. These results offer insights into how topology shapes dynamics, how dynamics learns topology and how topology evolves dynamically. This Perspective primarily aims to guide physicists, mathematicians, computer scientists and network scientists through the emerging field of higher-order topological dynamics, while also outlining future research challenges.

Understanding, modelling and predicting the emergent behaviour of complex systems are among the biggest challenges in current scientific research. Major examples include brain function, epidemic spreading and climate change. Through the use of graphs and networks to represent interactions, network science has provided a powerful theoretical framework that has deeply transformed the theory of complex systems. Networks encode relevant information about the complex systems that they represent[1,2], and their statistical and combinatorial properties strongly affect the unfolding of dynamical processes and critical phenomena[3–6]. The success of network science stems from the simplicity of its basic assumption: a complex system can be described in terms of interactions between its elements. However, this assumption also highlights a limitation of conventional network representations, as they encode only pairwise interactions.

As a matter of fact, representing a complex system with just pairwise interactions provides only an approximation of reality. Assuming the presence of many-body interactions is certainly more appropriate for many, if not all, systems. In high-energy physics,

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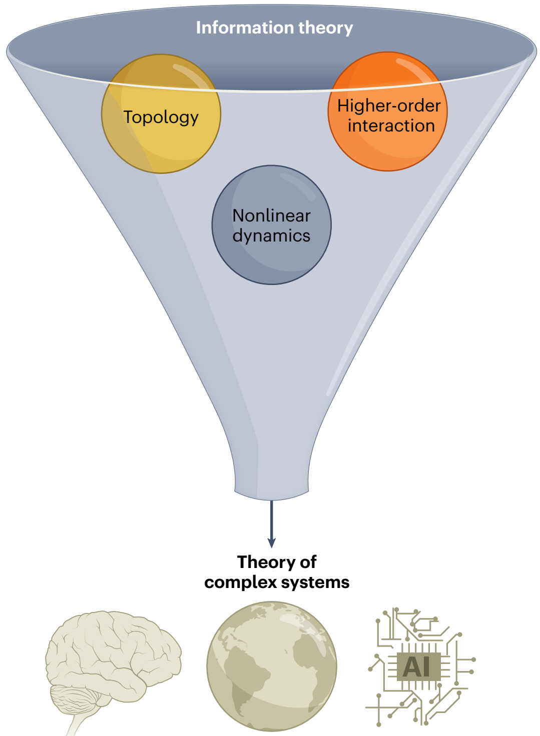

Fig. 1 | The emerging field of higher-order topological dynamics of complex systems. This field combines higher-order interactions, topology and nonlinear dynamics, giving rise to emergent phenomena encoding information that can dramatically transform our understanding of complex systems, such as the brain and the climate, and can allow the formulation of new efficient AI algorithms inspired by physics.

vertices of Feynman diagrams involve the creation and annihilation of more than two particles, with quantum chromodynamics notably admitting four-gluon vertices. Moreover, quantum many-body wave functions display strong higher-order quantum correlations and, owing to entanglement, cannot be fully described by two-point correlation functions. In inferential problems, a general multivariate distribution must involve higher-order interactions (as in higher-order graphical models, for instance). Likewise, in network science, pairwise networks cannot capture the many-body interactions that are present in the brain[7–12], social networks[13–16], ecosystems[17] and inferential financial models[18,19].

Allowing higher-order interactions leads to the formulation of higher-order networks that include interactions between two or more nodes[20–27]. Building blocks such as triangles, tetrahedra or hypercubes form the backbone of higher-order networks, yielding a topological description of complex systems that significantly alters our understanding of the interplay between structure and dynamics.

Topology involves the study of shapes and their invariant properties, such as Betti numbers and Euler characteristics. It plays a core role in topological data analysis[8,28–32], an approach for analysing the shape of high-dimensional and noisy data, and in one of its main tools: persistent homology. In particular, topological data analysis has already been shown to be key to detecting higher-order aspects of brain networks[7–12], offering a very powerful set of tools to characterize different states of brain activity. Topology has also been instrumental in the development of topological filtering algorithms[18,19], particularly for the analysis of financial data. Higher-order network structure can be investigated under the lens of topological[33] and homological percolation[34,35], the latter characterizing the emergence of large cycles and of higher-dimensional holes. Finally, by assigning weights to the higher-order interactions, weighted homology and

BOX 1

The topological spinor

Each topological signal defined on simplices of dimension n is an n-cochain C^n. If the simplicial complex is composed of N_0 nodes, N_1 edges and N_2 triangles, its dynamics is encoded by a topological spinor $\Psi \in C^0 \oplus C^1 \oplus C^2$ (ref. 89) (Fig. 2), which can be represented as the vector $\boldsymbol{\Psi} = (\boldsymbol{\theta}^\top, \boldsymbol{\phi}^\top, \boldsymbol{\xi}^\top)^\top$, where $\boldsymbol{\theta} \in \mathbb{R}^{N_0}$, $\boldsymbol{\phi} \in \mathbb{R}^{N_1}$ and $\boldsymbol{\xi} \in \mathbb{R}^{N_2}$ are topological signals defined on nodes, edges and triangles, respectively. The natural operators acting on topological signals are the boundary operators $B_{[n]}$ (ref. 78). On an unweighted network, $B_{[1]}$ represents the divergence, its transpose $B_{[1]}^\top$ represents the gradient and $B_{[2]}^\top$ represents the curl. The Hodge Laplacians $L_{[n]}$ (refs. 78,79) are defined through the boundary operators and describe diffusion from n simplices to n simplices through $n-1$ or $n+1$ simplices. Hence, the 1-Hodge Laplacian describes diffusion from edges to edges passing through either nodes or triangles. The boundary operators and the Hodge Laplacians are crucial to uncovering the interplay between structure and dynamics of higher-order networks. The key topological and spectral properties of $L_{[n]}$ are that the dimension of its kernel is given by the nth Betti number β_n and that its harmonic eigenvectors can be chosen on a basis in which they are mostly localized along n-dimensional holes. Moreover, the Hodge Laplacians obey Hodge decomposition, which implies that for an edge signal there is a unique way to decompose topological signals into a sum of an irrotational (curl-free) component, a solenoidal (divergence-free) component and a harmonic component. For a discussion of higher-order diffusion and random walks using the Hodge Laplacians and the boundary operators, see refs. 37,38,40,81. Note that the properties of this type of diffusion process are distinct from those defined on hypergraphs[107,108], as diffusion on simplicial complexes is permitted to go at most one dimension up or one dimension down.

cohomology have been shown to enhance the representation power of simplicial complexes, which can encode hypergraph data without loss of information[36].

Topology is crucial not only to characterize the structure of complex systems, but also to capture their higher-order dynamics. For the purpose of this Perspective, we are particularly interested in the emergent field of topological dynamics of higher-order networks, which combines topology with nonlinear dynamics. Specifically, higher-order topology unlocks fundamental mechanisms for higher-order topological diffusion[37–40], higher-order topological synchronization[41–47], topological pattern formation[48,49], triadic percolation[50–52], triadic neural networks[53,54] and topological machine learning algorithms[55–59]. These phenomena may be key to transforming our understanding of complex phenomena in neuroscience and climate change, and to formulating a new generation of physics-inspired machine learning algorithms (Fig. 1).

This Perspective focuses on the shift that the adoption of higher-order networks implies for the description of the interplay between topology and dynamics in complex systems. We outline key results and recent developments in the field, and the open challenges that future research must address. Accompanying supporting code, movies and material can be found in ref. 60.

Beyond the node-centric view of network dynamics

Dynamical systems defined on network topologies have received extensive scientific attention[3,4], but most works implicitly assume that

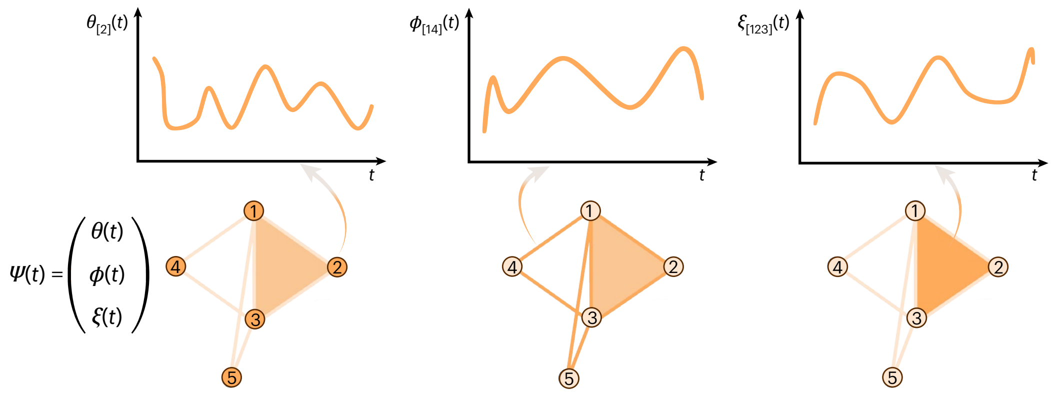

Fig. 2 | The dynamical state of a higher-order network. Going beyond the node-centred view of network dynamics, the dynamics of a higher-order network is captured by a topological spinor $\boldsymbol{\Psi}$, assigning a dynamical variable to each node, edge, triangle and higher-order simplices. The figure schematically shows the time series of the topological signal $\theta_{[2]}(t)$ of node [2], $\phi_{[14]}$ of edge [14] and $\xi_{[123]}(t)$ of triangle [123] as a function of time, t. Figure adapted with permission from ref. 48, APS.

the dynamical state of a network is defined exclusively by variables associated with its nodes. Although this is valid in some cases, such as in epidemic spreading, it generally represents a limiting assumption. For example, dynamical variables associated with the edges of a network, such as fluxes, are common.

In this regard, there is growing interest in topological signals—that is, dynamical variables associated not only with nodes, but also with edges, triangles or other higher-order structures. Examples include synaptic signals between neurons or edge signals at the level of brain regions[10,11] and general biological transportation networks[61]. Other instances of topological signals are currents at different locations in the ocean[40], the influence of volcanic activity on teleconnections in the climate[62] and the velocity of winds at a given altitude and geographical location.

Abstracting from these examples, the core idea is that the dynamical state of a simplicial complex formed by nodes, edges and triangles comprises topological signals defined on each dimension. These are encoded in the topological spinor (see Box 1 for a mathematical description and Fig. 2).

Whereas dynamical variables associated with edges of plaquettes are a widely accepted concept in other fields of physics, such as gauge theory[63] and quantum information[64], in network science and machine learning the description of the dynamical state of complex systems with a topological spinor represents a perspective that leads to new research questions. One prominent theme is the exploration of new algorithms for treating and processing topological signal data and the development of topological neural network architectures for predicting topological signals[55,58,65]. Current research is also building on the powerful tools provided by discrete topology and discrete exterior calculus[66] to characterize the collective behaviour of topological signals[41,42,48,49,67].

Topological synchronization

Synchronization[68–71] refers to the emergence of a collective and ordered dynamical motion of an extensive number of oscillators. Synchronization is universal in characterizing the dynamics of complex natural and man-made systems, ranging from brain dynamics to power grids and Josephson junctions. The two most fundamental models that capture the synchronization transition on complex networks are the Kuramoto model[68] and global synchronization[72,73] of coupled identical oscillators. Both approaches are traditionally defined for node topological signals; that is, oscillators located on the nodes of the network and coupled via edges. Whereas the Kuramoto model describes both global and partial synchronization of heterogeneous oscillators (that is, oscillators that in the absence of interactions exhibit different phases), global synchronization involves identical oscillators that may follow arbitrary, and even chaotic, dynamics.

Going beyond conventional dyadic networks, higher-order topological synchronization allows us to treat not only the synchronization of node topological signals, but also that of higher-order structures including edge topological signals—which are of particular interest for a variety of applications. Approaches to study higher-order topological synchronization include the topological Kuramoto model[41,67] and topological global synchronization[42], which display a rich phenomenology arising from the interplay of topology and dynamics.

The topological Kuramoto model (Box 2) in particular reveals a surprising connection with topology, showing that topology shapes the dynamics of higher-order networks, but also that the dynamics learns the underlying network topology. In fact, the topological Kuramoto model displays striking differences from the traditional node-based Kuramoto model. First, the synchronization dynamics of the n-dimensional topological signal is only possible if the higher-order network has at least one n-dimensional hole. Second, the synchronized state is localized on the holes of the higher-order structure. If the topological Kuramoto model is defined on a simplicial or cell complex that contains more than a single n-dimensional hole, the synchronization dynamics might be driven by a harmonic eigenvector localized on a single hole or by a linear combination of the harmonic eigenvectors localized on different holes (Fig. 3). Changing the homology of the simplicial complex by filling some holes, modulating their geometry by changing the metric matrices (weights associated with nodes, edges and higher-order simplices) and changing the intrinsic random frequencies of oscillators can affect the nature of the synchronized state. Research has found that higher-dimensional holes capture important dynamical information in the brain[7–9,12]. Therefore, whether the localized dynamics on the holes of a simplicial complex can be used to store information[74], and whether control theory[75] can provide key mechanisms for driving the dynamics towards a specific hole, or from one hole to another, are interesting research questions.

The higher-order topological Kuramoto dynamics (defined in equation (1) of Box 2) entails one linear transformation of the signal induced by a boundary operator and a nonlinear transformation due to the application of the sine function, concatenated by a second linear transformation induced by the boundary operator. These dynamical transformations also form the basis of simplicial neural architectures[57,76], especially in the case of weighted boundary matrices[66]. The interplay between topology and dynamics can be further enriched by considering weighted and directed versions of this model[44,45].

The study of global topological synchronization[42] has revealed useful insights, such as the conditions that allow topological oscillators sitting on higher-order simplices to globally synchronize.

BOX 2

The higher-order topological Kuramoto model

The higher-order topological Kuramoto model[41] captures the topological synchronization of the n-dimensional topological signal $\boldsymbol{\phi}$ of elements ϕ_α, describing non-identical oscillators placed on n-dimensional simplices. In this model $\boldsymbol{\phi}$ follows the dynamical equation:

$$\frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = \boldsymbol{\omega} - \sigma B_{[n]}^{\top} \sin\left(B_{[n]}\boldsymbol{\phi}\right) - \sigma B_{[n+1]} \sin\left(B_{[n+1]}^{\top}\boldsymbol{\phi}\right), \qquad (1)$$

where the sine function is taken element-wise, $B_{[n]}$ are the boundary matrices, σ is the coupling constant and $\boldsymbol{\omega}$ is the vector of intrinsic frequencies drawn from a random unimodal distribution, typically a Gaussian or Lorentzian distribution. The topological Kuramoto model reduces for $n=0$ to the standard node-based Kuramoto model[68]. It leads to a continuous synchronization transition for any $n \geq 0$, while the adaptive modulation of its coupling constants with the global order parameters gives rise to explosive discontinuous transitions[41,67]. The topological Kuramoto model is a gradient flow of the Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = -\boldsymbol{\omega}^{\top}\boldsymbol{\phi} - \sigma \mathbf{1}_{N_{n-1}}^{\top} \cos\left(B_{[n]}\boldsymbol{\phi}\right) - \sigma \mathbf{1}_{N_{n+1}}^{\top} \cos\left(B_{[n+1]}^{\top}\boldsymbol{\phi}\right), \qquad (2)$$

where $\mathbf{1}_{N_n}$ is the N_n column vector whose elements are all ones. Neglecting for the moment the term that depends on the random intrinsic frequencies, this Hamiltonian has a degenerate fundamental state with the degeneracy equal to β_n that includes all the eigenvectors in the kernel of $L_{[n]}$. The topological Kuramoto dynamics can be associated with global topological complex order parameters[41] given by $X_{\pm} = \sum_\alpha e^{i\phi_\alpha^{\pm}}/N_{n\pm1}$, where $\boldsymbol{\phi}^{+} = B_{[n+1]}^{\top}\boldsymbol{\phi}$ and $\boldsymbol{\phi}^{-} = B_{[n]}\boldsymbol{\phi}$. These order parameters indicate when the non-harmonic modes freeze. In Fig. 3 we indicate instead some local complex order parameters associated with the two octagons of the figure and given by $X_o = \sum_{\alpha\in\mathcal{O}} e^{i(-1)^{f(\alpha)}\phi_\alpha}/N_o$, where the sum is extended to all the edges α incident to the octagon $\mathcal{O}$, formed by $N_o=8$ edges, and where $f(\alpha)=0$ ($f(\alpha)=1$) if α is oriented clockwise (anticlockwise). As is apparent from Fig. 3, the order parameters X_o can help us distinguish between filled and unfilled cycles by considering only the topological dynamics of the edge signals, thus dynamics learns topology.

For node-based oscillators, the global synchronized state always exists. Its existence is ensured by the fact that the constant eigenvector is always a harmonic eigenvector of the graph Laplacian. As such, a basic question is whether the global synchronized state is dynamically stable. Moving to global topological synchronization of oscillators placed on higher-order topological signals, the existence of a globally synchronized state is not in general guaranteed. Moreover, the synchronized dynamics will be localized on the harmonic eigenvectors of dimension n. As a result, if the simplicial complex does not contain a harmonic eigenvector constant on all n-dimensional simplices, it is not possible to observe global topological synchronization of order $n > 0$; thus only some simplicial and cell complexes can sustain global synchronization. However, some cell complexes such as the square lattice tessellation of the torus can sustain global synchronization for topological signals of any order n (Fig. 3) and an appropriate choice of the weights of the simplicial complex can further facilitate the global synchronized state[77].

Hodge Laplacians[78,79] offer an alternative way to revisit higher-order diffusion on simplicial complexes, providing an extension for the notion of the spectral dimension[38,80] and a definition of higher-order random walks[40]. This allows a separation of the diffusion of the irrotational and solenoidal components of the dynamics[81] and control[37]. Hodge Laplacians also provide a spectral principle for community detection[82] related to clique communities[83] and k-connectedness[20].

The topological Dirac operator and higher-order dynamics

Although the Hodge Laplacians are suitable to treat topological signals of a given dimension (for example, edge signals or triangle signals), they fall short when describing how topological signals can crosstalk. Conversely, the topological Dirac operator (D, Box 3) enables the treatment of topological signals of different dimensions simultaneously and coherently, and it has recently emerged as a versatile algebraic operator with wide-ranging applications in the study of complex systems. Here we provide a few illustrative examples of its use in the field of higher-order topological dynamics.

Originally defined in lattice gauge theory to define staggered[63] and Dirac–Kälher fermions on lattices[84], the Dirac operator has since been adopted in the context of non-commutative geometry[85], in quantum graphs[86] and in quantum computation[87,88]. It has only been understood very recently that the topological Dirac operator is not only relevant to to quantum physics, but also has important applications in the study of complex systems and the collective phenomena that involve topological signals of different dimensions[89].

The topological Dirac operator is rooted in theoretical physics[89–92] and can be used to formulate a topological Dirac equation[89] in which the eigenstates have a topological interpretation and are defined on nodes, as well as links and higher-order simplices. For this equation, the matter–antimatter symmetry is broken for states of particles and antiparticles of energy $E = m$ (Fig. 4). Moreover, the topological Dirac operator can be used to define the mass[90] of simple and higher-order networks and is at the basis of an information theory coupling matter with geometry[92]. The topological Dirac operator can be coupled to metric matrices leading to symmetric and asymmetric weighted Dirac operators[36]. Moreover, as for the continuous Dirac operator, the topological Dirac operator can be coupled to gamma matrices, allowing it to act on generalized topological spinors comprising (for instance) two-dimensional (2D) node signals and 2D edge signals—or even a 2D node signal and 1D edge signals[49,89].

The topological Dirac operator naturally defines Dirac synchronization[43,46,93], a Kuramoto-like synchronization model and global topological Dirac synchronization[47] locally coupling node and edge topological signals. It is worth noting that in Dirac synchronization the Dirac coupling induces a discontinuous synchronization transition on fully connected networks, as well as random networks. One peculiar property of Dirac synchronization is that the order parameter of the dynamics includes linear combinations of node and edge signals. Moreover, Dirac synchronization leads to the emergence of spontaneous rhythms; that is, fluctuations of the order parameter. This is a promising tool for the study of biological rhythms in the brain and rhythmic behaviour in the climate system—such as the see-saw relationship of the East Asian–Australian summer monsoon[94].

The topological Dirac operator is also key to defining topological patterns that extend the notion of Turing patterns in the continuum[95] or on networks[96,97] to topological signals defined on nodes, edges and squares of cell complexes[48,49]. Interestingly, patterns formed by one-node and one-edge signals can only be static. However, by using the three-way Dirac operator acting on 2D-node and 1D-edge topological signals (or, as a matter of fact, 1D-node and 2D-edge topological signals), Dirac patterns with very distinct dynamical signatures emerge[49].

The spectral properties of the topological Dirac operator encode topological features of the simplicial complexes, and its weighted

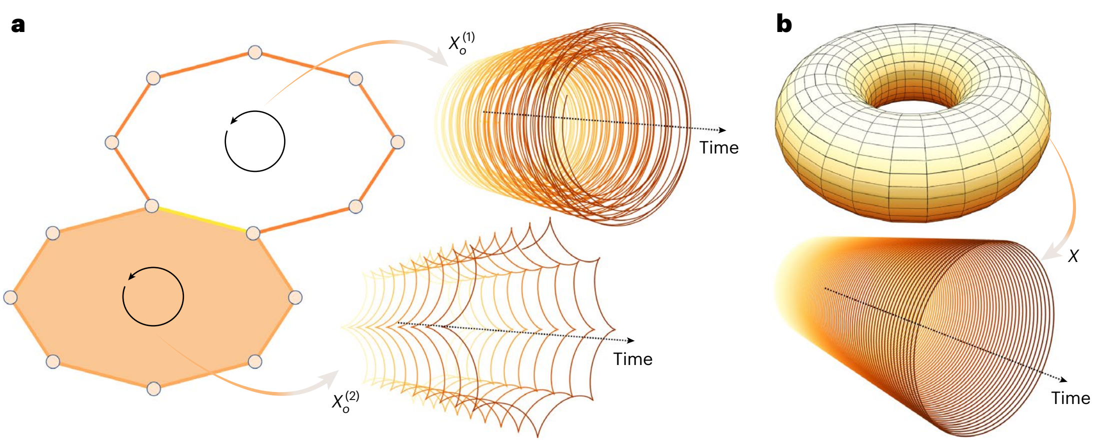

Fig. 3 | The topological Kuramoto model and global synchronization. The synchronization of the topological signals is driven by the presence of n-dimensional holes in the higher-order network. **a**, In the topological Kuramoto model[41], empty and full cells (the case of edge synchronization for a cell complex with one empty cell or hole is shown) display different types of dynamical states localized on the two cycles of the higher-order network, as is evident from the dynamics of the local complex order parameters. $X_o^{(1)}$ and $X_o^{(2)}$ defined in Box 2. In the limit in which there are no n-dimensional holes, the dynamics of the n-order topological Kuramoto model freezes. **b**, While the higher-order topological Kuramoto model can also achieve synchronization if the holes of the cell complex are localized, global synchronization requires the existence of a harmonic eigenvector constant on each cell of the cell complex, which is achieved if there is a single delocalized hole, such as for the square lattice tessellation of the torus show here, displaying a global complex order parameter $X = \sum_\alpha e^{i\phi_\alpha}/N_n$ oscillating in phase while keeping its absolute value $|X| = 1$.

BOX 3

The topological Dirac operator

On an unweighted simplicial complex of dimension 2, the topological Dirac operator[89] D: $C^0 \oplus C^1 \oplus C^2 \to C^0 \oplus C^1 \oplus C^2$ is given by

$$D = \begin{pmatrix} 0 & B_{[1]} & 0 \\ B_{[1]}^{\top} & 0 & B_{[2]} \\ 0 & B_{[2]}^{\top} & 0 \end{pmatrix} \quad \text{with} \quad D^2 = \mathcal{L} = \begin{pmatrix} L_{[0]} & 0 & 0 \\ 0 & L_{[1]} & 0 \\ 0 & 0 & L_{[2]} \end{pmatrix}. \tag{3}$$

Hence, the D can be interpreted as the 'square root' of the Laplacian and admits both positive and negative eigenvectors related by chirality. Note, however, that the harmonic eigenvectors break the chiral symmetry. On a 2D simplicial complex, D maps $\mathbf{\Psi}$ into the topological spinor $\mathbf{\Phi}$ given by:

$$\mathbf{\Phi} = D\mathbf{\Psi} = \begin{pmatrix} B_{[1]}\boldsymbol{\phi} \\ B_{[1]}^{\top}\boldsymbol{\theta} + B_{[2]}\boldsymbol{\xi} \\ B_{[2]}^{\top}\boldsymbol{\phi} \end{pmatrix}, \tag{4}$$

by projecting topological signals one dimension up or down, thus allowing them to crosstalk. D can be adopted to define a topological Dirac equation[89] given by:

$$\mathrm{i}\partial_t \mathbf{\Psi} = (D + \gamma_0 m)\mathbf{\Psi}, \tag{5}$$

where, for a 2D simplicial or cell complex, γ_0 is the $\mathcal{N} \times \mathcal{N}$ block diagonal matrix with diagonal blocks $(I_{N_0}, -I_{N_1}, I_{N_2})$ (Fig. 4).

version reveals the geometric degree of freedom of the simplicial complex. Moreover, it can be defined for single and multiplex networks[98]. The strong interplay between the structure of the simplicial complex and the spectral properties of the topological Dirac operator is at the core of its growing popularity in persistent homology[87,88,99,100]. Owing to its ability to treat coupled topological signals of different dimensions, the topological Dirac operator is also used for signal processing on simplicial complexes and hypergraphs[56,101] and for formulating Gaussian kernels[102] and simplicial neural networks[103,104].

The topological Dirac operator, which in its continuum version has played such a central role in different areas of physics, is therefore now emerging as a fundamental algebraic topology tool for the study of complex systems. Despite being in its infancy, we believe that research in this field has great potential for discoveries in complex systems, as well as for progress in machine learning algorithms rooted in (and inspired by) physics.

Topology is dynamical

So far, we have only discussed models in which the topology of the simplicial complexes is independent of time. However, many complex systems exhibit situations where topology changes in time, as is typically observed in brain resting activity and climate systems. Basic motifs of such time-evolving networks are triadic interactions, occurring when nodes regulate the interactions among other pairs of nodes. Triadic interactions can be signed; that is, the regulator node can either enhance or inhibit the interactions between other two nodes. A network with triadic interactions is at the same time a higher-order network and a network of networks. Indeed, triadic interactions include more than two nodes and can be combined such that a single interaction is regulated by more than one node. Typical examples are neuron–glia interactions in the brain, transcription networks and ecological interactions[17]. Although triadic interactions are a well-established concept in each of these scientific domains, it has only recently become clear that their role in modulating interactions leads to new dynamical processes, including triadic percolation[50–52], the triadic Hopfield model[53], more general neuron–astrocyte models of associative memory[54] and information propagation in non-equilibrium signalling networks[105].

Triadic interactions can be used to regulate on or off structural edges, leading to triadic percolation[50,52]—a fully-fledged dynamical process in which the giant component changes through time. Specifically, triadic percolation implements a two-step algorithm: (1) nodes are active when they belong to the giant component of the structural network formed by nodes and active structural edges, and (2) active nodes regulate the structural edges through their associated triadic interactions. The resulting percolation process has an order parameter (given by the fraction of nodes in the giant component) that

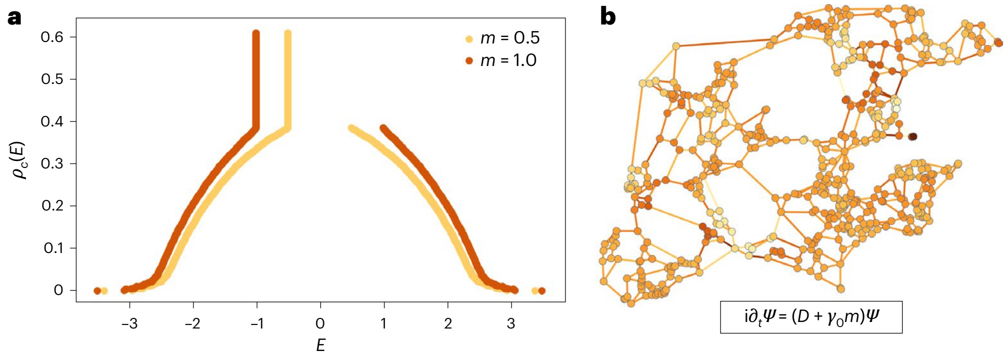

Fig. 4 | The properties of the topological Dirac operator and the topological Dirac equation. **a**, Spectra of topological Dirac equations with masses $m = 1/2$ and $m = 1$ of the fungal network shown in **b**. **b**, The eigenstate of the topological Dirac equation with $E = -1.1929$ and $m = 1$, indicated by the different colours of the nodes and edges of the network. In **a** the spectra are described by the cumulative distribution of energy states $\rho_c(E)$ indicating the number of eigenstates of energy E' of the topological Dirac equation with $E' > E$ for positive values of E and $E' < E$ for negative values of E. It is apparent that the spectra are symmetric, with the only exception being eigenstates of energy E with $|E| = m$ that are not chiral. Data for the fungi network are from Pp_M_Tokyo_U_N_26h_6 in ref. 106.

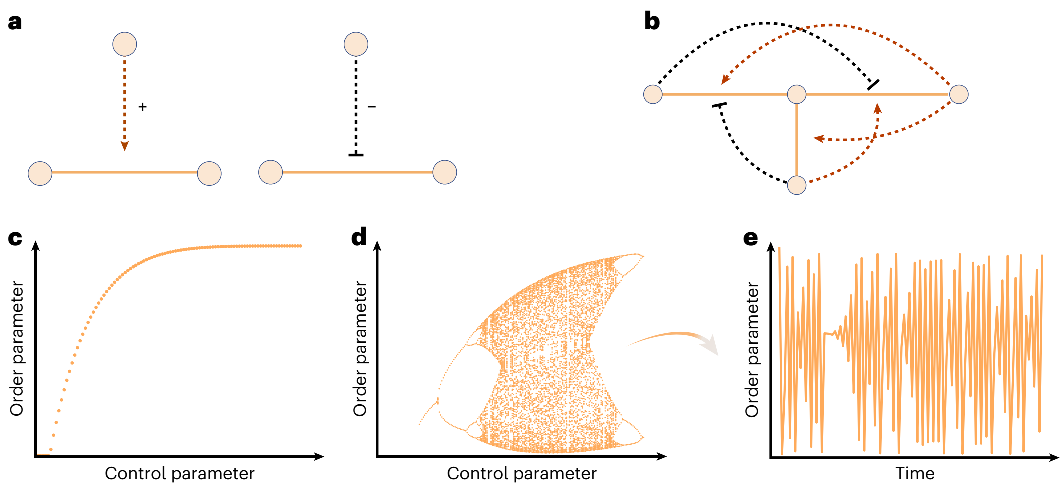

Fig. 5 | Signed triadic interactions and the phase diagram of triadic percolation. **a**,**b**, Triadic interactions occur when one node regulates the interactions between two other nodes either positively or negatively (**a**), leading to higher-order networks with triadic interactions (**b**). The inclusion of triadic interactions turns percolation into a fully fledged dynamical process that allows the temporal modulation of the giant component. **c**,**d**, The phase diagram of percolation (**c**), which shows a second-order phase transition in the absence of triadic interactions, can be predicted to be an orbit diagram (**d**) when triadic interactions take place, demonstrating that the percolation order parameter undergoes a route to chaos. **e**, Thus, for some control parameter values, the order parameter changes in time following a chaotic time series.

is time-varying and can be proved to undergo a route to chaos in the universality class of the logistic map. Therefore, the phase diagram, instead of describing the standard second-order percolation transition, is represented by an orbit diagram. This reveals that the percolating phase is characterized by non-stationary dynamics of the order parameter (Fig. 5). Thus the network can display a blinking behaviour (that is, periodic activation of different sets of nodes) or, in the infinite network limit, a chaotic dynamics of the order parameter. In spatial networks defined on a 2D torus, triadic percolation[51] gives rise to complex spatiotemporal patterns and to a giant component whose topology changes through time and displays (for some parameter values) intermittency between patterns of different topologies. These results provide new scenarios for understanding the spatiotemporal modulation of the giant component in neuroscience and climate science, for example.

Outlook

Higher-order networks are starting to reveal how network topology, which is key in traditional physics fields such as high-energy physics and condensed and soft matter, is also crucial to capture higher-order network dynamics. This emerging theoretical framework discloses innovative findings in various disciplines ranging from physics, computer science, Earth science and neuroscience to finance. In this Perspective we have covered fundamental aspects of this nascent field of research, and the first interdisciplinary applications, by outlining the key challenges that emerge from the results obtained so far.

The growing field of higher-order topological dynamics offers interesting prospects for the development of a more comprehensive theory of complexity by combining higher-order networks with topology and nonlinear processes, and by interpreting these interactions under the lens of information theory.

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Author contributions

G.B. designed the project. A.P.M., H.S., L.G., R.M., T.C., J.J.T. and G.B. prepared the figures and wrote the codes. All authors wrote the manuscript.

Competing interests

The authors declare no competing interests.

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